Learning Goals

1. Definitions of Universal Domain, Pareto Optimality, Non-Dictatorship, Independence from Irrelevant Alternatives.
2. Arrow's Impossibility Theorem.
3. Monotonicity Criterion.

Topic 5: A Perfect Voting System

We saw that there are drawbacks to each type of voting system we studied. The plurality method with three or more candidates may lead to a winner who is undesirable to the majority of voters. We saw that a Condorcet winner may not be the winner using the Borda method and in fact may not even make it into the runoff in a Plurality voting system with runoff. Furthermore a Condorcet winner may not even exist.

It is natural to try to find a perfect (or good) voting system. In order to explore the possibilities, we must first define what we mean by perfect or good. We follow the ideas of Nobel Laureate Kenneth Arrow who, beginning in the 1940's explored methods of ordering choices among public policies. We first begin with a list of properties that most people would consider desirable in a voting system.

Universal Domain Any ordering of the candidates is allowed, that is, there are no restrictions placed on the ranking of the candidates a voter may choose.

Pareto optimality If all voters prefer candidate A to candidate B, then the group choice should not prefer candidate B to candidate A.

Non-Dictatorship No one individual voter preference totally determines the group choice.

Independence from irrelevant alternatives If a group of voters choose candidate A over candidate B, then the addition or subtraction of other candidates should not change the group choice to B.
This requirement is the most debatable. On the one hand a choice between A and B should not depend on what other choices are available, on the other hand however, it is only by comparison with other possibilities that voters' perception of differences between candidates can be brought to light.

Example Plurality with runoff method violates the independence from irrelevant alternatives condition:

#Voters	4	3
A	1	2
B	2	1

#Voters	2	2	3
A	2	1	2
B	3	2	1
C	1	3	3

We see in the example on the left above that 4 out of 7 voters prefer A to B and that A is destined to win. On the right we introduce a third candidate, C, and two of those who initially voted A first, now vote for C as number 1, but still prefer A to B. We see that B now wins, which means that the introduction of the irrelevant alternative C reversed the outcome.

Arrow's Impossibility Theorem There is no voting system based on **rankings** that satisfies the properties of universal domain, Pareto optimality, non-dictatorship and independence from irrelevant alternatives.

It is important to note here that Arrows theorem applies in the context of rankings, where voters give an ordering of their preferences. It does not apply to the situation where voters give a measure of the worth or utility or strength of performance of each candidate (as with median range voting discussed in the article on voting in Oscars or current voting of judges in Gymnastics or ice-skating). Basically it says that it is impossible to find a function or rule that will amalgamate a sequence of individual rankings or ballots (represented by lists or lists with ties) in a reasonable way. This is of course an issue if one wants to amalgamate sports rankings, a problem we will have to contend with in our March Madness project. We also note that there are many other desirable properties of voting which are not listed above and should be considered when choosing a way to amalgamate ballots.

Example The method of pairwise comparisons (equivalent to Copeland's method) violates the independence from irrelevant alternatives criterion.
Suppose an NFL team will be getting the number one choice in the upcoming draft of college football players. After narrowing the list of candidates to five players (Allen, Byers, Castillo, Dixon and Evans), the coaches and team executives meet to discuss the candidates and eventually have a vote, a decision of major importance to both the team and the chosen player. According to team rules, the final decision must be made using the method of pairwise comparisons(Candidates are compared in pairs by a head-to-head comparison, getting 1 point for a win, 1/2 pt. for a draw and 0 points for a loss). The table below shows the preferences of the voters.

#Voters	**2**	**6**	**4**	**1**	**1**	**4**	**4**
A	1	2	2	3	3	2	5
B	4	1	1	2	4	5	4
C	3	3	5	1	1	4	2
D	2	4	3	4	2	1	3
E	5	5	4	5	5	3	1

The ten possible pairwise comparisons give

A vs. B −> 7 to 15 −> B: 1 pt.
A vs. C −> 16 to 6 −> A: 1 pt.
A vs. D −> 13 to 9 −> A: 1 pt.
A vs. E −> 18 to 4 −> A: 1 pt.
B vs. C −> 10 to 12 −> C: 1 pt.
B vs. D −> 11 to 11 −> B: 1/2 pt., D: 1/2 pt.
B vs. E −> 14 to 8 −> B: 1 pt.
C vs. D −> 12 to 10 −> C: 1 pt.
C vs. E −> 10 to 12 −> E: 1 pt.
D vs. E −> 18 to 4 −> D: 1 pt.

Results: A: 3 pts. B: 2.5 pts. C : 2 pts. D: 1.5 pts. E: 1 pt.

Now right before the draft, one of the players (Castillo) had accepted a scholarship to go to medical school and will not be playing professional football. Since Castillo was not top choice, this fact should have no effect on the choice of Allen as the first draft, or should it ?

Eliminate Castillo from the original election and relabel the voters' preferences 1-4, keeping the original order of preference for each voter. Using the new table of preferences, perform pairwise comparisons for candidates A, B, D and E and decide which player wins in the new scenario.

#Voters	**2**	**6**	**4**	**1**	**1**	**4**	**4**
A	1	2	2	2	2	2	4
B	3	1	1	1	3	4	3
D	2	3	3	3	1	1	2
E	4	4	4	4	4	3	1

A vs. B −> 7 to 15 −> B: 1 pt.
A vs. D −> 13 to 9 −> A: 1 pt.
A vs. E −> 18 to 4 −> A: 1 pt.
B vs. D −> 11 to 11 −> B: 1/2 pt., D: 1/2 pt.
B vs. E −> 14 to 8 −> B: 1 pt.

D vs. E $->$ 18 to 4 $->$ D: 1 pt.

The Monotonicity Criterion

Variations of Arrow's theorem using other criteria of fairness can be stated. Another desirable criterion frequently cited is the monotonicity criterion.

The Monotonicity Criterion If choice A is the winner of an election and, in a reelection, the only changes in the ballots are changes that only favor A, then A should remain the winner of the election.

Below we give an example showing how the plurality with runoff method violates the monotonicity criterion.

Example Three cities Athens (A), Babylon (B) and Carthage (C), are competing to host the next summer Olympic games. The final decision is made by a secret vote of the 29 members of the Executive Council of the International Olympic Committee, and the winner is chosen using the plurality with runoff method. Two days before the election is to be held, a straw vote is conducted by the Executive Council just to see how things stand. The results of the **straw poll** are shown below.

#Voters	**7**	**8**	**10**	**4**
A	1	3	2	1
B	2	1	3	3
C	3	2	1	2

The **results of the straw vote** are as follows:
Athens 11, Babylon 8, Carthage 10.
Babylon gets eliminated first and Carthage picks up 8 votes making the results of the second round
Athens 11, Carthage 18.
Thus Carthage wins.

Now although the results of the straw vote are supposed to be secret, word gets out that unless some of the voters turn against Carthage, Carthage will win. Because everybody wants to be on the winning side, what happens in the actual election is that even **more first place votes are are cast for Carthage than in the straw poll.** Specifically, **the four voters in the last column above decide as a block to switch their first place votes from Athens to Carthage**. This can only help Carthage, right?

The actual election results are as follows:

#Voters	**7**	**8**	**14**
A	1	3	2
B	2	1	3
C	3	2	1

Apply the plurality with runoff method to determine the winner.

Note Although there is no perfect voting system, a lot of current research is devoted to finding good voting sytems, such as systems which satisfy weaker conditions or systems which satisfy a set of strong criteria a large proportion of the time.

Sports Scoring Systems and Irrelevant Alternatives

Although Arrow's Theorem does not always apply to scoring systems in sport because they are not always systems that amalgamate individual preferences, one can still talk about desirable properties of scoring systems as we did about methods of amalgamating ballots. In particular, one would expect that a scoring system in any competition would satisfy the independence from irrelevant alternatives criterion. If, for example, A is ranked higher than B then the addition or subtraction of other competitors should not result in B being ranked higher than (or equal to) A. We look at two examples which clearly violate this criterion.

Cross Country Running In a cross country race, a standard team typically consists of seven runners. A team's score is the sum of the placings of its first 5 runners. Teams are ranked in order of their scores from lowest to highest. Although the sixth and seventh runners on a team do not contribute to the score of a team, they can increase the final score of other teams. In a paper on the subject on the MAA website http://www.mathaware.org/mam/2010/essays/ , Stephen Szydlik considers the results of a meet at Wisconsin-LaCrosse where 33 teams and 223 team runners raced over a 5 mile course. The final places for each team were:
First Place: UW-Madison runners finished in places 1, 2, 3, 8, and 27, Score: 41
Second Place: UW-LaCrosse runners finished in places 4, 12, 15, 24, 35, 49 and 55 , Score: 90
Third Place: UW-Oshkosh runners finished in places 10, 11, 13, 28, 30, 43 and 69. Score: 92

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
M_1	M_2	M_3	L_1	?	?	?	M_4	?	O_1	O_2	L_2	O_3	?	L_3					

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
?	?	?	L_4	?	?	M_5	O_4	?	O_5	?	?	?	?	L_5

(a) Szdlik points out that in a head to head comparison between UW-Oshkosh and UW-LaCrosse, UW-Oshkosh would have won. Removing all other teams from the race the runners from these two teams would have come in in the following order:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
L_1	O_1	O_2	L_2	O_3	L_3	L_4	O_4	O_5	L_5	O_6	L_6	L_7	O_7

where L_1 means the first runner from UW-LaCrosse etc...
UW-LaCrosse Score : $1 + 4 + 6 + 7 + 10 = 28$
UW-Oshkosh Score : $2 + 3 + 5 + 8 + 9 = 27$

The Ice Skating Paradox [1] The final results for the Ladies' singles in Ice Skating from the Winter Olympics of 2002 are shown below.

Full Results [edit]

Rank	Name	Nation	Points	SP	FS
1	Sarah Hughes	United States	3.0	4	1
2	Irina Slutskaya	Russia	3.0	2	2
3	Michelle Kwan	United States	3.5	1	3
4	Sasha Cohen	United States	5.5	3	4
5	Fumie Suguri	Japan	8.5	7	5
6	Maria Butyrskaya	Russia	8.5	5	6
7	Jennifer Robinson	Canada	11.0	8	7
8	Júlia Sebestyén	Hungary	11.0	6	8
9	Viktoria Volchkova	Russia	16.0	12	10
10	Silvia Fontana	Italy	17.5	11	12
11	Elina Kettunen	Finland	18.0	18	9
12	Galina Maniachenko	Ukraine	18.5	15	11
13	Sarah Meier	Switzerland	20.5	9	16
14	Elena Liashenko	Ukraine	21.0	16	13
15	Laetitia Hubert	France	22.0	14	15
16	Vanessa Gusmeroli	France	22.0	10	17
17	Yoshie Onda	Japan	22.5	17	14
18	Julia Soldatova	Belarus	29.0	22	18
19	Idora Hegel	Croatia	30.5	23	19
20	Vanessa Giunchi	Italy	30.5	21	20
21	Zuzana Babiaková	Slovakia	31.0	20	21
22	Mojca Kopač	Slovenia	31.5	19	22
23	Roxana Luca	Romania	35.0	24	23
WD	Tatiana Malinina	Uzbekistan		13	
Free Skate Not Reached					
25	Stephanie Zhang	Australia		25	
26	Park Bit-Na	South Korea		26	
27	Julia Lebedeva	Armenia		27	

There are two parts to the competition, the short program and the long program. The winner of the short program was awarded 0.5 points, the next skater 1 point, the next 1.5 points and the fourth 2 points. At the end of the short program the results were

Kwan (0.5), Slutskaya (1.0), Cohn (1.5), Hughes (2.0).

For the long program, the winner was awarded 1 point, the next skater 2 points, the next 3 points and the fourth 4 points. When the long program is finished, each skaters scores for both events are added and the skater with the lowest score wins the gold medal.

After Hughes, Kwan and Cohen had skated in the long program, Hughes was leading with a long program score of 1, Kwan was second with a long program score of 2 and Cohen was third with a long program score of 3. Thus before Slutskaya skated, the totals were

Kwan (2.5), Hughes (3.0), Cohn (4.5).

When Slutskaya skated, she was placed second in the long program and the scores for the long program were now

Hughes (1), Slutskaya (2), Kwan (3), Cohn (4).

The final result was now

Hughes (3.0), Slutskaya (3.0), Kwan (3.5), Cohn (5.5).

Hughes' superior performance in the Long Program was used to break the tie giving Hughes the gold medal.

We see that Kwan was ahead of Hughes before Slutskaya skated, but after Slutskaya skated she found herself behind Hughes. The scoring system has now changed to a range voting system which does satisfy Pareto Optimality and Independentce from Irrelevant Alternatives and non-dictatorship. It is not a system based on ranking, rather the athletes are given scores which do not change when an athlete is added or removed from the field. (Note that when an athlete is removed from or added to the field in both of the above examples, the places or ranks of the remaining athletes can change.)

[1] 100 essential things you didn't know you didn't know about sport, *John Barrow*

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And the Oscar Goes to...Not Its Voting System

Selection of Academy Award Nominees and Winners is Flawed, but Reformers Can't Seem to Elect a Better Candidate

By CARL BIALIK

Academy Award nominees and winners are selected using two different voting systems that are, according to some political mathematicians, the worst way to convert voters' preferences into an election outcome.

The nominees are selected using a system called instant runoff, which has been adopted in some municipal and state elections. Out of last year's 281 eligible films, each voter selects five nominees in order of preference for, say, best picture. All movies without any first-place votes are eliminated. The votes for those films with the least first-place votes are re-assigned until five nominees have enough.

One problem with that system is a kind of squeaky-wheel phenomenon: A movie that is second place on every ballot will lose out to one that ranks first on only 20% of ballots but is hated by everyone else. Then, in another upside-down outcome, a movie can win for best picture even if 79% of voters hated it so long as they split their votes evenly among the losing films. This isn't as unfamiliar as it sounds: Some people think Al Gore would have won the Electoral College in 2000 if Ralph Nader hadn't diverted more votes from him than he took from former President George W. Bush.

"It's crazy," says Michel Balinski, professor of research at École Polytechnique in Palaiseau, France. The nomination system's properties are "truly perverse and antithetical to the idea of democracy," says Steven Brams, professor of politics at New York University. He thinks the final vote for the Oscar winner may be even worse than the selection of nominees.

The big problem: If voting systems themselves were put to a vote, prominent scholars would each produce a different ballot, then disagree about which system should be used to select the winner. So it's no surprise that advocates of alternate voting systems, which range from simple yes/no approval ratings to assigning numerical scores to each candidate, have had little

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more luck reforming political elections than they have with entertainment awards.

Consider two systems that, on the surface, seem similar. Prof. Balinski and mathematician Rida Laraki have devised a system they call majority judgment that requires voters to rank each candidate on a scale from 1 to 6. The votes are lined up in order, and each candidate is assigned the middle, or median, score. The highest median score wins. Another system, range voting, isn't that different: The candidate with the highest average, or mean, score wins.

Yet the second system's leading advocate, Warren D. Smith, co-founder of the Center for Range Voting, has devoted a Web page to the Balinksi-Laraki system's "numerous disadvantages."

Brace yourselves for "Ishtar" defeating "The Godfather." Suppose 49 voters award "The Godfather" six points and "Ishtar" only four. One voter grants the desert debacle four points and the mafia masterpiece three, and the remaining 49 award "The Godfather" three points and "Ishtar" only one point. "Ishtar" actually wins with a median score of four points compared to "The Godfather's" three points. Prof. Balinski, in turn, calls range voting a "ridiculous method," because it can be manipulated by strategic voters.

Despite the flaws in Oscars voting, the system remains as it has since 1936. Every 15 years or so, the Academy re-examines its voting and has decided to stick with it, says the Academy of Motion Picture Arts and Sciences' executive director, Bruce Davis. "It is a very effective method of reflecting the will of the entire electorate," Mr. Davis says.

Choosing a Winner | Conducting and deciding a vote using an instant runoff

STAGE 1

Voters are asked to rank the candidates 1–4.

After the ranking, no candidate has a majority, but A has the lead

Voters	A (Leader)	B	C	D
1	1	4	3	2
2	1	4	3	2
3	1	4	3	2
4	1	4	3	2
5	1	4	3	2
6	4	1	3	2
7	4	1	3	2
8	4	1	3	2
9	4	1	3	2
10	4	2	1	3
11	4	2	1	3
12	3	4	2	1
13	3	4	2	1
14	4	3	1	2

STAGE 2

D has the fewest first-place votes

So D's votes get reassigned to the second choice on those ballots, C.

Now C has five No. 1 rankings, so A and C are tied.

A (Tied)	B	C
1	4	3
1	4	3
1	4	3
1	4	3
1	4	3
4	1	3
4	1	3
4	1	3
4	1	3
4	2	1
4	2	1
3	4	1
3	4	1
4	3	1

STAGE 3

Now B has fewest votes.

B's votes get reassigned to C, because C was the third choice of the B voters and the second choice, D, has been eliminated.

C beats A, seven votes to four.

A	C (Winner)
1	3
1	3
1	3
1	3
1	3
4	1
4	1
4	1
4	1
4	1
4	1
3	1
3	1
4	1

But many voting theorists aren't so keen on the system. It's called instant runoff because it is used in political elections in lieu of a two-stage vote in which top candidates compete again if none receives a majority of the vote. Among the potential problems, showing up to vote for your favorite candidate may create a worse outcome than not showing up at all. For example, your vote could change the order in which candidates are eliminated, and the next-in-line candidate on the ballot for the newly eliminated film may be a film you loathe.

To choose Oscar winners, voters simply choose their favorite from the nominees, and the contender with the most votes wins. That could favor a film that has a devoted faction of fans, and sink films with overlapping followings who split their vote. Even most critics of instant runoff say it beats this plurality system that led to the Gore-Nader-Bush result. In the film realm, Prof. Brams of NYU blames the current system for the best-picture victory of "Rocky" over films such as "Network" and "Taxi Driver" that he speculates would have won head to head.

How this works out in reality is hard to know, because the Academy doesn't release any details about the balloting, even after the telecast, in part to avoid shaming fifth-place films. Mr. Davis says even he never learns the numbers from his accountants: "Are there years when I'm curious as to what the order of finish was? Absolutely. But I recognize it as a vulgar curiosity in myself."

More

The Oscars involve two stages of voting, for nominees and for winners. Delve into the math of elections in **the Numbers Guy blog**.

Complete Coverage: Academy Awards

Such secrecy frustrates voting theorists who are anxious for experimental data about voter behavior that may help them choose from among different voting systems. Without such evidence, they are left to devise their own studies, to dream up examples that sink rival systems or to create computer simulations to study how easily different systems can be manipulated.

Sports fans cry manipulation when votes don't go as they'd hoped. Many sports awards and rankings are derived from what is known as Borda count, which asks voters to rank candidates and then assigns points on a sliding scale, with the most for first-place votes and the least for last-place ones.

Critics of these systems fear that strategic voters will assign their top choice the highest possible score, and everyone else zero, thereby seizing more power than voters who approach the system earnestly; or, in the case of rankings, bury or omit a preferred candidate's top rival. Boston Red Sox fans will tell you to this day that such strategic voting by a New York beat writer cost Pedro Martinez the American League Most Valuable Player award a decade ago.

Says Prof. Balinksi, "Not everyone will do it, but enough will do it to manipulate the results."

There is a philosophical question obscured by that criticism: Should voters with stronger feelings have more influence? A voter may support Candidate A strongly and loathe all the rest; two other voters may like Candidate A but slightly prefer B. Should B beat A even though all voters would have been fine with A?

Some scholars back the Condorcet winner, the candidate that would beat all others in head-to-head matchups. Trouble is, there isn't always one. As an alternative, Prof. Brams advocates approval voting, which tallies the number of voters who approve of each candidate and chooses the one with the most votes.

Rob Richie, executive director of FairVote, which has had success pushing the adoption of instant runoff for elections, says that approval voting doesn't fly with politicians: They're uncomfortable with the idea that voters who prefer them might throw equal support to a rival. For advocates of alternate systems, it's crucial to get support from politicians because voters aren't likely to get excited about such issues unless the country is hanging on a chad.

Mr. Richie argues that, in practice, instant runoff hasn't displayed the feared paradoxes. He says his critics should go get their preferred systems adopted so they can offer their own proofs of concept. He adds that mathematicians haven't made much headway changing voting laws "so they hound reformers who are being successful, and that's just irritating."

Vanderbilt University mathematician Paul H. Edelman, who has consulted with the Country Music Association on its annual awards, says his colleagues should tone down the dogma and embrace a range of voting systems for different situations. "The mistake that mathematicians make is to assume that all elections are the same," Prof. Edelman says. "That's a terrible thing to do."

Get Me a Recount

While Academy Award nominees and winners are selected using two different voting systems, there are at least six other major ones that have been proposed and studied by scholars. And each one can produce different outcomes from the same ballots.

In a hypothetical 11-voter election, in which voters score eight candidates from 0 to 20, each candidate would win under one of eight major voting systems. Bolds mean that voter approves that candidate -- roughly equivalent to a yes/no vote.

Number of Ballots	Candidate A	Candidate B	Candidate C	Candidate D	Candidate E	Candidate F	Candidate G	Candidate H
4	**18**	4	5	17	15	0	13	14
3	0	**14**	5	**11**	**12**	**10**	8	9
2	0	**12**	**20**	**10**	**11**	**9**	**18**	**19**
1	2	0	**12**	**17**	1	**11**	**16**	3
1	0	1	4	2	3	**16**	**15**	5
Wins in	**Plurality**	**Runoff**	**Instant runoff**	**Borda count**	**Condorcet**	**Approval voting**	**Mean range voting**	**Median range voting**

See how each candidate wins in each system:

A wins in plurality: A has four first-place votes, more than any other candidate.

B wins in runoff: All but the top two first-place vote getters, A and B, are eliminated. B is preferred by three of the four voters who ranked other candidates first, and beats A, 6-5.

C wins in instant runoff: Under this system, each voter selects five nominees, in order, in a given category. E, G and H have no first-place votes and are eliminated first. Then come D and F, which each have one first-place vote. Among remaining candidates, C ranks second on those ballots, so C picks up two more first-place votes and is now tied with A, with four. B, with three, is eliminated next, and C ranks above A on the ballots that belonged to B, so C beats A, 7-4.

D wins in Borda count: Borda count asks voters to rank candidates and then assigns points on a sliding scale, with the most for first-place votes and the least for last-place ones. On each ballot, give seven votes to the first-place contender, six to second, and so on, down to zero for the last-place candidate. D edges E, 52-48.

E wins in Condorcet: The Condorcet winner is the candidate which beats all others in head-to-head matchups. E beats every other candidate head to head, by ranking higher than each on a majority of ballots. E beats A, 6-5; B, 6-5; C, 6-5; D, 6-5; F, 9-2; G, 7-4; H, 7-4.

F wins in approval voting: This system tallies the number of voters who approve of each candidate and chooses the one with the most votes. F is approved by seven voters, edging D, approved by 6.

G wins in mean range voting: The mean vote for G is 13, edging D, with 12.7.

H wins in median range voting: The median vote for H is 14, beating G, which has 13.

Sources: Center for Range Voting; WSJ Research

Write to Carl Bialik at numbersguy@wsj.com

Corrections & Amplifications
Warren Smith is co-founder of the Center for Range Voting. He is no longer affiliated with Temple University. A previous version of this column incorrectly referred to him as a Temple mathematician. In addition, a label is incorrect in the graphic accompanying this column. In the final stage of the runoff, C beats A, 9-5, not 7-4.

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